PERIODIC MOTIONS OF A VIBRATING STRIKER INCLUDING A SLIPPAGE REGION

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A novel type of periodic motions in which a part of the trajectory is composed of an infinite sequence of collision-collisionless segments, is investigated for the systems with collisions. The problem of existence in the dynamic systems with collision interactions of such a sequence was studied in [1]. (*)
In analogy with other piecewise continuous systems, this sequence of motions is called "slippage". The case of an infinite sequence of collisions occurring in a dynamic model of a clock movement was studied in [2], while the case arising in the process of impulsive braking with dry friction was studied in [3] under the name of quasi-plastic collision.

The problem of constructing the boundaries of the slippage region in the phase space is solved as well as that of defining the region of existence of stable periodic motions with a slippage region, in the parametric space of a single-mass vibrating striker.

The results of numerical computations are given for the oscillations occurring in the presence of an external periodic force, which are of greatest practical importance.

Numerous investigations of the dynamic models of the systems with impulsive interactions have shown the existence of modes of varying complexity, the complexity determined by the ratio of the periodicity of the motion to the periodicity of the driving force and by the number of impacts per single period of motion. It was found that the increase in complexity is accompanied by appreciable narrowing of the regions of existence and stability of the mode in the parametric space.

The regions of existence of stable periodic motions with an infinite convergent sequence of collisions defined in the parametric space of the vibrating striker, and of a simplest system containing two colliding pairs [4], were found to be of the same order as the corresponding regions of the one-impact modes. This indicates the practical importance of the motions with slippage for the systems with collisions. The presence of such motions makes possible an interpretation of a number of experimental results within the framework of the Newton's hypothesis. Such would be e.g. the finite duration of impact of a vibrohammer when the velocity restitution coefficient is different from zero [5 - 7] or the

*) The proof given in [1] and valid for the case in which the collisions are not accompanied by jumps in acceleration can easily be extended to the case when such changes exist and are caused by the linear frictional forces [4].

manner in which the escapement of a marine chronometer interacts with a shock bearing, the interaction consisting of the first, not completely elastic collision followed by a second inelastic collision, with subsequent motion under a kinematic constraint [8].

1. Construction of the boundaries of the slippage region in the phase space. The model of a vibrating striker adopted here consists of a mass m attached to a spring with a linear characteristic k and subjected to the action of an external perturbation defined as a sum of $F \sin\omega t$ and a constant force P. The displacement x of the mass m is restricted by a fixed barrier and on making a contact with the barrier the mass undergoes an instantaneous collision with the velocity restitution coefficient being equal to R. When the elastic constraint is not deformed, the gap between the mass m and the fixed barrier is characterized by the parameter D.

Starting with the usual assumptions [9] and choosing $y = F^{-1}\omega^2 mx$ and $\tau = \omega t$ as the dimensionless variables, we arrive at the following equations:
The equations of collisionless motion

$$y^{\cdot\cdot} + \lambda^2 y = -\lambda^2 d + \sin\tau, \qquad y > 0 \tag{1.1}$$

The collision interactions

$$y_+^{\cdot} = -Ry_-^{\cdot}, \qquad y = 0 \tag{1.2}$$

and the equation describing a possible state of kinematic constraint between the mass and the barrier

$$y = y^{\cdot} = 0, \qquad \sin\tau - \lambda^2 d < 0 \tag{1.3}$$

Here the position of the phase point at the surface Π of the collision interactions is taken as the origin $y = 0$, the symbols $y_-^{\cdot}$ and $y_+^{\cdot}$ denote the precollision and postcollision velocities of the mass m The dimensionless eigenfrequency λ of the vibrating striker and the gap d are expressed in terms of the initial parameters by

$$\lambda^2 = km^{-1}\omega^{-2}, \qquad d = -F^{-1}(m\omega^2 D + P\lambda^{-2})$$

Thus the behavior of the system (1.1) - (1.3) depends on three real parameters λ, d and R varying over the limits

$$0 \leqslant \lambda < \infty, \qquad |d| < \infty, \qquad 0 \leqslant R < 1$$

The phase space of this system formed by the coordinates y, $y^{\cdot}$, τ is threedimensional. According to [1], the region Π_s, which we call the slippage plate, is situated on the surface Π near the boundary

$$y = 0, \qquad y^{\cdot} = 0, \qquad y^{\cdot\cdot} < 0$$

After the representative point $M(y, y^{\cdot}, \tau)$ has arrived at Π_s, its further motion takes place along the phase trajectory consisting of an infinite sequence of collision-collisionless intervals and terminates at the "point of convergence" M_0 whose coordinates are found from the conditions

$$y = y^{\cdot} = y^{\cdot\cdot} = 0, \qquad y^{\cdot\cdot\cdot} > 0$$

For the present mathematical model of the vibrating striker the coordinates of the point of convergence are

$$y_0 = 0, \qquad y_0^{\cdot} = 0, \qquad \tau_0 = \arcsin\lambda^2 d \tag{1.4}$$

During the motion that follows, the representative point moving along the trajectory emerging from M_0 leaves the region Π_s and enters the half-space $y > 0$. Let us denote by T a point transformation which maps the surface Π onto itself and is generated by two consecutive parts of the motion, the collision interaction and the collisionless motion. When $i \to \infty$, the point of convergence M_0 is the limit of the convergent sequence of transformations T^i of the region Π_s into itself. The problem of establishing the exact boundary of the slippage plate consists of finding a limit set in the neighborhood of which on one side we find only the points which are transformed into M_0 by T^i, and no such points on the other side. The required boundary of Π_s consists of the negative part of the $y^{\cdot\cdot}$-axis and of a curve W emerging from M_0 and situated in the region $y = 0,\ y^{\cdot} < 0,\ y^{\cdot\cdot} < 0$. [1]. This suggests a simple method of obtaining the boundary curve W, based on iterating the inverse point transformations T^{-i} It is sufficient to construct a sequence of approximations $W_i = T^{-i}\ (W_0)$, for which

$$W = \lim_{i\to\infty} T^{-i}\ (W_0)$$

Here W_0 is the initial approximation. The process W_i converges to the limiting set W from the inside of Π_s, if $W_0 \subseteq \Pi_s$ and from the outside, when $W_0 \not\subseteq \Pi_s$. We note that a segment of an arbitrary curve emerging from M_0 and situated within the half-space $y = 0,\ y^{\cdot} < 0$ can serve as the initial approximation W_0

The magnitude of the accelerations in the system (1.1) to (1.3) remains unchanged during the collisions. We can therefore prescribe the initial approximation $W_0 \subseteq \Pi_s$ using the conditions ensuring that the curve [1]

$$0 < Y_* < y^{\cdot\cdot\cdot}(\tau) < Y, \qquad \frac{9R-5}{12R} < \frac{Y_*}{Y}$$

$$0 < -y^{\cdot\cdot}(\tau) < \frac{2Y\tau_*}{3-\sqrt{4-5R}}, \qquad 0 < \frac{-y^{\cdot}(\tau)}{[y^{\cdot\cdot}(\tau)]^2} < \frac{5(1-R)}{24RY}$$

lies on the slippage plate. At the point of convergence (1.4) the quantity $y^{\cdot\cdot\cdot}(\tau) = \cos\tau$ and is therefore bounded from above by the value $Y = 1$. The solution $y(\tau)$ of the Eq. (1.1) describing the collisionless motion is written in terms of functions whose Taylor expansions have unbounded radii of convergence τ_*. As a result we obtain the initial approximation in the form

$$y^{\cdot} = -\frac{5(1-R)}{24R}(\lambda^2 d - \sin\tau)^2$$

$$-\frac{\pi}{2} < \tau < \arcsin\lambda^2 d, \qquad \cos\tau > \frac{9R-5}{12R}$$

On the other hand, we can use the segment

$$y = 0, \qquad y^{\cdot} = 0, \qquad y^{\cdot\cdot} > 0$$

of the half-axis adjacent to the point of convergence as the intial approximation W_0 With such an initial approximation we can determine the slippage region by constructing a sequence of iterations W_i disposed outside the plate Π_s.

The equations of the point transformation T^{-i} mapping Π into itself and defined by the system of equations (1.1) and (1.2), is written in the form

$$\lambda^{-1}(y_i^{\cdot} - \alpha\cos\tau_i)\sin\lambda(\tau_i - \tau_{i-1}) + d + \\ + (\alpha\sin\tau_i - d)\cos\lambda(\tau_i - \tau_{i-1}) - \alpha\sin\tau_{i-1} = 0 \qquad (1.5)$$

$$\lambda\,(\alpha \sin \tau_{i-1} - d)\ \sin \lambda\,(\tau_i - \tau_{i-1}) - y^{\cdot}_i +$$
$$+\ \alpha \cos \tau_i - (Ry^{\cdot}_{i-1} + \alpha \cos \tau_{i-1}) \cos\ \lambda\,(\tau_i - \tau_{i-1}) = 0 \qquad (1.6)$$
$$\alpha = (\lambda^2 - 1)^{-1}$$

Using the recurrent relations (1.5) and (1.6) (which can also be studied using the method developed in [11]) we can find the coordinates of the point M_{i-1} provided that the coordinates of the preceding point M_i are already known. Specifying successively the point M_i on the curves W_0, W_1, $W_2, \ldots$, we obtain the points M_{i-1} belonging to the curves W_1, $W_2, W_3, \ldots$

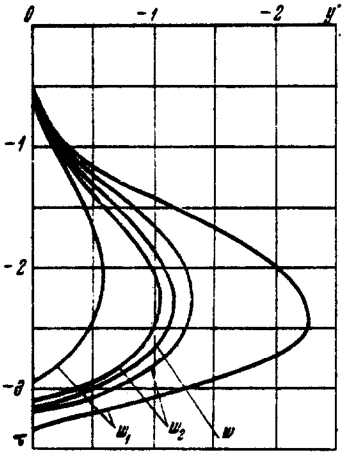

Fig. 1.

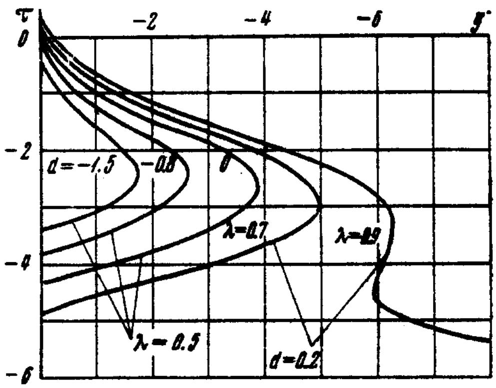

Fig. 2.

Figure 1 depicts the results of computing a sequence of iterations W_i for $R = 0.2$, $\lambda = 0.3$ and $d = -5.3$. The family of the limit sets W shown in Fig. 2 was constructed for $R = 0.2$ and various values of λ and d. In all cases two processes of consecutive approximations W_i, one situated inside and the other outside the region Π_s ,were considered. We note that for $i > 3$ the curves W_i of both sequences become practically identical. Equations (1.5) and (1.6) indicate that the slippage plate grows in size without bounds in the negative direction of the $y^{\cdot}$-axis as $R \to 0$.

2. Construction of the region of existence of stable periodic motions with slippage in the parametric space. The slippage mode terminates at the point of convergence. The motion which follows is realized along the phase trajectory of collisionless motions emerging from M_0 , until some instant $\tau_1' > \tau_0$ corresponding to the successive arrival of the trajectory at the point M_1 on Π The relations determining the motion of the vibrating striker during the time interval $\tau_0 \leqslant \tau < \tau_1$ are obtained from the solution of (1.1) with the initial conditions coinciding with the coordinates of the point of convergence (1.4). In consequence we have the following system of equations determining the coordinates of the point M_1 $(0, y_1^{\cdot}, \tau_1)$:

$$\lambda\,[d - \alpha \sin \tau_1 + (\alpha \sin \tau_0 - d) \cos \lambda\,(\tau_1 - \tau_0)] +$$
$$+\ \alpha \cos \tau_0 \sin \lambda\,(\tau_1 - \tau_0) = 0 \qquad (2.1)$$
$$\lambda\,(d - \alpha \sin \tau_0)\ \sin \lambda\,(\tau_1 - \tau_0) + y_1^{\cdot} +$$
$$+\ \alpha\,[\cos\ \tau_0\ \cos\lambda\ (\tau_1 - \tau_0)\ - \cos \tau_1] = 0 \qquad (2.2)$$

Here τ_1 is the smallest root of (2.1) satisfying condition $\tau_1 > \tau_0$.

If the phase point M_1 lies within Π_s, then the subsequent motion will again be, for a certain period of time, a slippage motion. The period $2\pi n$ of such simplest forced oscillations with slippage region is defined as the time elapsed between two consecutive passages of the phase point through M_0, and $n = 1,2,3,...$ denotes the ratio of the period of the forced oscillations to the period of the external harmonic force.

The region of existence of periodic motions with slippage is situated between two surfaces in the parametric space. One of these surfaces denoted by C_s corresponds to the degeneration of periodic motions containing an infinite sequence of collisions into a mode containing a finite number of collisions per period, while the other bifurcation surface is defined as the boundary of the region of existence of real values of τ_0, the region defined by the inequality $|\lambda^2 d| \leqslant 1$. Obviously, the surface C_s represents the combinations of the values of the parameters λ, d and R, which ensure that the point M_1 belongs to the corresponding boundary set W (λ, d, R). Thus the properties of the branching surfaces shown above enable us to determine the region of existence of the periodic motions with slippage in the parametric space of the dynamic system under consideration. The corresponding forced oscillations are stable, since a set mapped into itself (the point of convergence) is of zero dimension.

The general approach to constructing the boundary surface C_s described above is reduced to performing some numerical scheme whose structure would lend itself to processing in the form of a machine algorithm. For this reason further studies were conducted with the help of a digital computer.

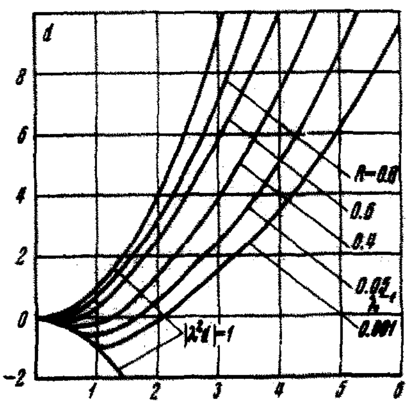

Fig. 3.

Figure 3 shows various sections D_{s1} of the region of existence of stable periodic motions with slippage produced by the planes R = const. The subscript 1 appearing in the notation of these intersections indicates that the frequency of the periods $n = 1$. For a given value of R the region D_{s1} is contained between the boundary $|\lambda^2 d| = 1$ and the corresponding curve belonging to the family shown on the figure.

The results obtained indicate that stable periodic motions with slippage can be realized for both, the positive and the negative values of the gap and for any value of the velocity restitution coefficient within $0 < R < 1$. When the latter value is small, the intervals of variation of λ and d become very large. Comparison with the results of [9] shows that when $R \to 0$, the boundary of D_{s1} transforms continuously into the boundary of the region of existence and stability of the simplest, one-collision oscillations with a halt. Thus, when R increases from zero in a sufficiently smooth manner, the one-collision oscillations with a halt transform into the oscillations with slippage region, giving rise to periodic motions with a finite number of collisions per period.

3. Discussion of results. Modelling. It is known that the behavior of the model considered within the framework of periodic motions with a finite number of collisions per period gives insufficient agreement with the experiment when the velocity restitution coefficient is small. The periodic motions observed experimentally in [5] and 6] are characterized by the fact that the duration of the collision interactions differs appreciably from zero and that the region of existence and stability in the parametric space is extended towards the large values of λ and is larger than predicted theoretically in [9]. This cannot be explained by considering the periodic motions with a finite number of collisions per period; the regions of existence of these motions narrow to such an extent that they fail to overlap the inevitable scatter of the parameters of the real structures. Even the assumption that the collision interactions are completely inelastic does not lead to a satisfactory computing scheme.

The present results indicate that the observed discrepancy between the theory and experiment can be explained in physically meaningful terms by the fact of existence of slippage. Indeed, when $R \to 0$, the slippage may occur in a region belonging to the phase space and increasing without bounds, and periodic motions with slippage can be realized in an appreciable part of the parametric space adjacent to the region of one-collision oscillations.

Although the duration of the slippage mode can, in general, be of considerable length, the duration of the separate intervals of the collisionless motion diminishes rapidly as the phase point approaches the point of convergence. For this reason, remembering that experiments have limited accuracy, we can naturally interpret the essential duration of the collision interactions [5 and 6], in accordance with the Newton's hypothesis, as an extension of the slippage mode. The experimentally observed widening of the regions of existence of one-collision periodic motions stems from the fact that the latter motions are difficult to distinguish from the periodic motions with slippage, when the values of the parameters fall near the boundary of C_s. The basic results obtained here were confirmed by investigating the equations (1.1) to (1.3) on an analog computer. Here we stipulated a switchover of the scheme given by [10] to modelling the motion of the kinematically constrained (by (1.3)) elements under the assumption that the relative post-collision velocities in the slippage mode are sufficiently small.

Fig. 4.

Figure 4 a, b, represents the oscillograms of the periodic motions with slippage for the following values of the parameters on D_{s1}

$$(a)\ R = 0.6, \quad d = 1.6, \quad \lambda = 1.5$$
$$(b)\ R = 0.15, \quad d = -0.01, \quad \lambda = 0.97$$

In conclusion we note that a simple approach in investigating the distinctive features of the existence of the slippage mode combined with computerized iteration of the corresponding point transformations make possible a sufficiently exhaustive study of the periodic motions containing an infinite sequence of collisions.

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